THE ARITHMETIC GEOMETRIC MEAN INEQUALITY

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Theorem 0.1. *(Arithmetic Geometric Mean Inequality) If $x_i \geq 0$ for $i = 1, \ldots, n$, then*

$$\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^{n} x_i$$

Proof. Let $x_i \geq 0$ for $i = 1, \ldots, n$. If $n = 1$, then the inequality is satisfied (with equality). So let's take $n \geq 2$. We consider two cases.

Case 1: There exists an $i \in \{1, \ldots, n\}$ with $x_i = 0$. Then $\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} = 0$. Because all x_i are non-negative, $\frac{1}{n}\sum_{i=1}^{n} x_i \geq 0$. This proves the inequality for this case.

Case 2: $x_i > 0$ for $i = 1, \ldots, n$. Define

$$s = \sum_{i=1}^{n} x_i$$

Now consider, for $s > 0$ fixed, the optimization problem

$$\max_{\mathbf{x} \in \mathcal{F}} \phi(\mathbf{x})$$

with

$$\phi(\mathbf{x}) = \left(\prod_{i=1}^{n-1} x_i\right)\left(s - \sum_{i=1}^{n-1} x_i\right)$$

$$\mathcal{F} = \left\{\mathbf{x} \in \mathbb{R}^{n-1} : x_i > 0 \text{ for } i = 1, \ldots, n-1 \text{ and } \sum_{i=1}^{n-1} x_i < s\right\}$$

Note that the closure of $\mathcal{F}$ and the boundary of that closure are given by

$$\overline{\mathcal{F}} = \left\{\mathbf{x} \in \mathbb{R}^{n-1} : x_i \geq 0 \text{ for } i = 1, \ldots, n-1 \text{ and } \sum_{i=1}^{n-1} x_i \leq s\right\}$$

$$\partial\overline{\mathcal{F}} = \left\{\mathbf{x} \in \mathbb{R}^{n-1} : x_i = 0 \text{ for an } i \in \{1, \ldots, n-1\} \text{ or } \sum_{i=1}^{n-1} x_i = s\right\}$$

See figure 0.1 for an example plot of ϕ.

ϕ is continuous and $\overline{\mathcal{F}}$ is closed and bounded, so ϕ has a maximum on $\overline{\mathcal{F}}$. We know that any global maximum of ϕ on $\overline{\mathcal{F}}$ lies either on the boundary $\partial\overline{\mathcal{F}}$ or in the interior, which equals $\mathcal{F}$. Note that

$$\phi(\mathbf{x}) = 0 \text{ for } \mathbf{x} \in \partial\overline{\mathcal{F}}$$

Any maximum of ϕ at $\mathbf{x} \in \mathcal{F}$ must be a critical point, i.e. it must satisfy

$$\nabla\phi(\mathbf{x}) = 0$$

For $i = 1, \ldots, n-1$

$$\frac{\partial}{\partial x_i}\phi(\mathbf{x}) = \left(\prod_{j=1, j\neq i}^{n-1} x_j\right)\left(s - \sum_{j=1}^{n-1} x_j - x_i\right)$$

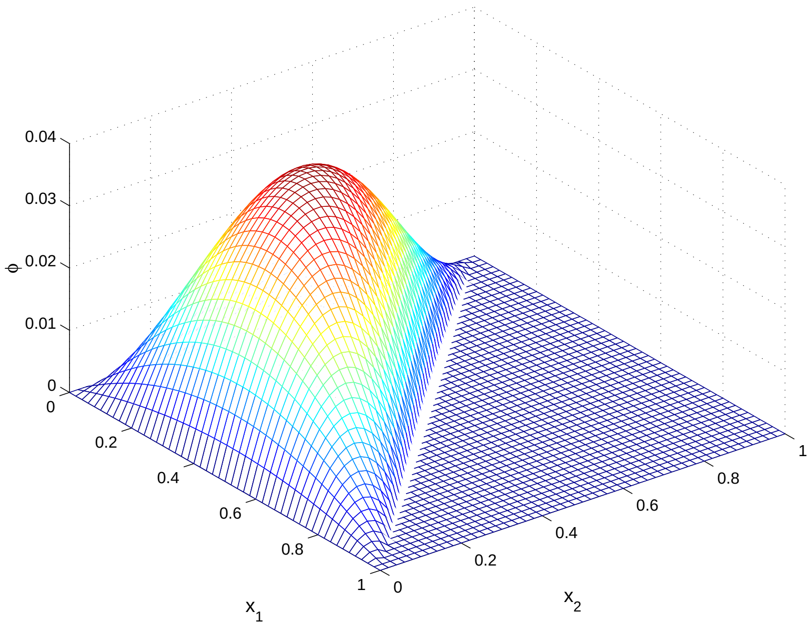

FIGURE 0.1. Example plot of ϕ with $n = 3$ and $s = 1$

Since $x_i \neq 0$ for $i = 1, \ldots, n-1$ on $\mathcal{F}$, $\mathbf{x}$ must satisfy

$$s - \sum_{j=1}^{n-1} x_j - x_i = 0 \text{ for } i = 1, \ldots, n-1$$

This be written as the system of linear equations

$$A\mathbf{x} = \begin{bmatrix} 2 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 1 & & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & & 1 & 2 & 1 \\ 1 & 1 & \cdots & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} s \\ s \\ \vdots \\ s \\ s \end{bmatrix}$$

which has the unique[1] solution

$$\hat{\mathbf{x}} = \left(\frac{s}{n}, \ldots, \frac{s}{n}\right)^T \in \mathcal{F}$$

Note that

$$\phi(\hat{\mathbf{x}}) = \left(\frac{s}{n}\right)^n > 0 = \phi(\mathbf{x}) \text{ for all } \mathbf{x} \in \partial\overline{\mathcal{F}}$$

We conclude that there is a unique global maximum of ϕ on $\overline{\mathcal{F}}$, at $\hat{\mathbf{x}}$.

[1]To see that a solution is unique, consider that, writing $A = [\mathbf{a}_1 \ldots \mathbf{a}_{n-1}]$ and $I = [\mathbf{e}_1 \ldots \mathbf{e}_{n-1}]$ for the identity matrix, we have

$$\mathbf{e}_i = -\frac{1}{2n}\mathbf{a}_1 - \ldots - \frac{1}{2n}\mathbf{a}_{i-1} + \frac{1}{2}\mathbf{a}_i - \frac{1}{2n}\mathbf{a}_{i+1} - \ldots - \frac{1}{2n}\mathbf{a}_{n-1}$$

for $i = 1, \ldots, n-1$. This shows that the matrix A is invertible, from which the statement follows.

Returning to our Case 2, we see that $(x_1, \ldots, x_{n-1}) \in \mathcal{F}$ and thus

$$\begin{aligned} \phi(x_1, \ldots, x_{n-1}) &\leq \phi\left(\frac{s}{n}, \ldots, \frac{s}{n}\right) \\ \prod_{i=1}^{n} x_i &\leq \left(\frac{s}{n}\right)^n \\ \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} &\leq \frac{s}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i \end{aligned}$$

This proves the inequality for this case. □