1 Periodic Tilings and Symmetry Groups

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1.1 Tiles and Tilings

Before saying anything about *periodic* tilings, let us first define what a *tiling* is. The following definition formalizes the concept that a tiling consists of tiles that together cover the whole plane, but without overlapping.

Definition 1.1. A plane *tiling* is a countable collection $\mathcal{T} = \{T_1, T_2, \ldots\}$ of closed sets in the Eucledian plane $(\mathbb{R}^2, \|\cdot\|_2)$ whose union is the entire plane and whose interiors are mutually disjoint:

$$\bigcup_{T \in \mathcal{T}} T = \mathbb{R}^2, \qquad \bigcup_{\substack{S,T \in \mathcal{T} \\ S \neq T}} S^\circ \cap T^\circ = \emptyset.$$

The sets $T \in \mathcal{T}$ are called the *tiles* of $\mathcal{T}$.

This definition could be extended to more dimensions, but here we will simply consider the plain plane, $\mathbb{R}^2$, equipped with the Eucledian norm $\|\cdot\|_2$ and its induced metric, or distance function, $\mathrm{d}(\cdot,\cdot)$. We will not give an axiomatic description of the Eucledian plane here — remember this is a course on tilings — but we will assume that you are familiar with basic geometry.

The above definition of a tiling is in fact a bit too general for most purposes, so often we put some additional restrictions on the tiles. An example condition is that all tiles are so called topological disks. The idea is that this rules out tiles that have holes, are not connected or (partly) consist of lines or curves.

In [GS], the following definition is given for a topological disk: a set $T \subset \mathbb{R}^2$ is called a *closed topological disk*[1] when there exists a homeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ that maps T onto the the closed unit disk $D := \{x \in \mathbb{R}^2 : \mathrm{d}(0,x) \leq 1\}$. A *homeomorphism* in turn is a bicontinuous bijective mapping. A bijective mapping ϕ is *bicontinuous* if both ϕ and ϕ^{-1} are continuous. Note that in the usual definition of a topological disk, ϕ needs to be a homeomorhpism from T do D, but [GS] defines it in this way. It makes it easy to proof the following lemma:

Lemma 1.2. *A topological disk has non-empty interior.*

Proof. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism that maps a tile T to the closed unit disk D. Denote the open unit disk by E, then $E \subset D$ and by the continuity of ϕ, the set $\phi^{-1}(E)$ is an open subset of $\mathbb{R}^2$. By the bijectivity of ϕ we have $\phi^{-1}(E) \subset T$ and $\phi^{-1}(E) \neq \emptyset$, hence T contains a non-empty open set, namely $\phi^{-1}(E)$. But then T has non-empty interior. □

1.2 Symmetries

Some tilings have a certain kind of 'symmetry' in themselves, namely that under a rigid motion of the plane, like a rotation or translation, the tiling remains 'the same'. We intend to formalize this concept so that we can classify tilings according to these symmetry properties. We start by defining the rigid motions we have in mind:

Definition 1.3. An *isometry* is a mapping $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ that preserves distances, i.e. for all $A, B \in \mathbb{R}^2$ we have $\mathrm{d}(\sigma A, \sigma B) = \mathrm{d}(A, B)$.

A special isometry is the *identity mapping*, denoted by Id, that simply maps every point to itself. The isometries of the plane are easily classified:

Theorem 1.4. *Any isometry $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ is*

1. *a* rotation *of an angle θ around a point C, denoted by $R_{C,\theta}$, or*

[1] An *open topological disk* must be homeomorphic with the *open* unit disk.

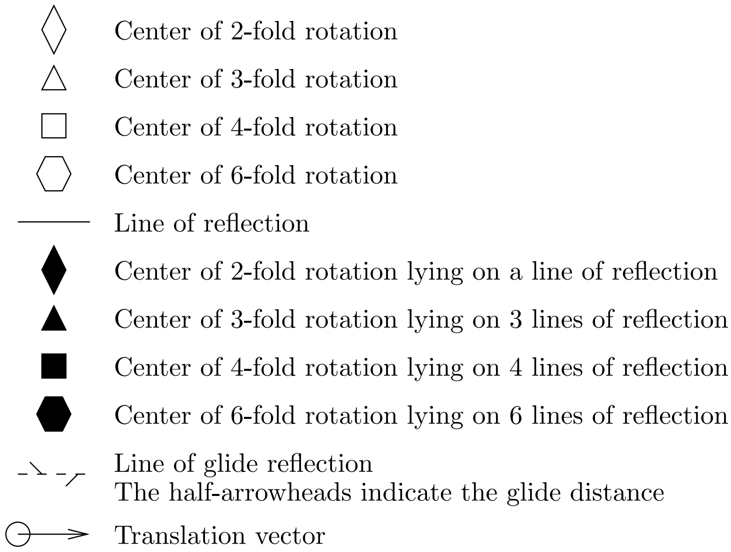

Figure 1: Summary of symmetry symbols commonly used.

2. a translation *with a vector* $\mathbf{u}$*, denoted by* $T_{\mathbf{u}}$*, or*

3. a reflection *in a line* L*, denoted by* M_L*, or*

4. a glide reflection*: a reflection in the line* L*, followed by a translation of distance* d *along that line, denoted by* $G_{L,d}$*.*

Before going to a sketch of the proof of this theorem, let us first make a few remarks. Note that the identity mapping Id could be classified as either a translation over the zero vector or a rotation over an angle that is an integer multiple of 2π. Also, a reflection could be seen as a glide reflection followed by a translation over zero distance. We also distinguish between *direct* and *indirect* isometries: direct isometries do *not* change the orientation of the plane, indirect isometries *do* change the orientation. With 'orientation' we mean e.g. the orientation of the three vertices of a triangle: clockwise or anti-clockwise. See Figure 1 for details on the used symbols to indicate symmetries in figures.

Sketch of proof of Theorem 1.4. Let $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ be an isometry. Look at the images under σ of the origin O and the two points A and B with respective coordinates $(1, 0)$ and $(0, 1)$. We can start by choosing σO anywhere in the plane. Then we must now choose σA somewhere on the circle with radius 1 and center σO. The same holds for σB, but now we are left with two only choices as we must have $\mathrm{d}(\sigma A, \sigma B) = \mathrm{d}(A, B) = \sqrt{2}$.

Now note that we have completely determined σ, as every point in the plane is uniquely determined by its distance to three points that are not on a straight line, and σO, σA and σB are three such points.

By distinguishing several cases for the choice of σO, σA and σB, the theorem can now be proved. If the orientation of the triangle OAB is not changed by σ, we can distinguish two cases:

- The triangle OAB is not rotated; then σ is simply a translation by σO (seen as vector).
- The triangle OAB is rotated; then σ is a rotation about some point in the plane; the contruction of this point we leave as an exercise.

If the orientation of the triangle OAB is changed by σ, we can contruct a line of glide-reflection. The details of this construction we also leave to the reader. If we find the associated distance of movement to be zero, we can simply call σ a reflection. □

We can now easily proof the following corollary:

Corollary 1.5. *Every isometry $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ is bijective.*

Proof. Let $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ be an isometry, then it is of one of the types described in Theorem 1.4, and we can give its inverse σ^{-1} depending on this type:

$$R_{C,\theta}{}^{-1} = R_{C,-\theta}, \qquad T_{\mathbf{u}}{}^{-1} = T_{-\mathbf{u}}, \qquad M_L{}^{-1} = M_L, \qquad G_{L,d}{}^{-1} = G_{L,-d},$$

for any line point C, angle θ, vector $\mathbf{u}$, line L and distance d. □

After having defined the 'rigids motions' mentioned in the introduction (as being isometries), we are ready for symmetries:

Definition 1.6. A *symmetry* of a *collection* $\mathcal{C}$ of sets $S \subset \mathbb{R}^2$ — like a tiling — is an isometry σ that maps every set $S \in \mathcal{C}$ onto some set in $\mathcal{C}$: $\forall S \in \mathcal{C}$ we have $\sigma S \in \mathcal{C}$. A symmetry of a *set* $S \subset \mathbb{R}^2$ is a symmetry of the collection $\mathcal{C} = \{S\}$, i.e. σ maps S onto itself: $\sigma S = S$.

We will denote the collection of all symmetries of a set $S \subset \mathbb{R}^2$ by $\mathcal{I}(S)$, and the collection of all symmetries of a collection $\mathcal{C}$ of subsets of the plane by $\mathcal{I}(\mathcal{C})$. Additionally, the set of all direct symmetries of a collection $\mathcal{C}$ we will denote by $\mathcal{I}(\mathcal{C})^+$, and the set of all indirect symmetries of a collection $\mathcal{C}$ by $\mathcal{I}(\mathcal{C})^-$. Note that all these sets of symmetries are in fact subsets of $\mathcal{I}(\mathbb{R}^2)$, whose elements we classified in Theorem 1.4.

Before continuing, we owe you the definition of a periodic tiling. Intuitively, we think of a periodic tiling as a tiling with a repeating pattern in two directions. We can now define it using the terminology just introduced:

Definition 1.7. A *periodic tiling* is a tiling $\mathcal{T}$ with two non-parallel translations over positive distance in its symmetries: there exist $T_{\mathbf{u}}, T_{\mathbf{v}} \in \mathcal{I}(\mathcal{T})$ with $\mathbf{u}$ and $\mathbf{v}$ linearly independent vectors.

1.3 Groups

We will see that the symmetries of a set or a tiling have an algebraic structure called a group, but not after having defined what a group actually is.

Definition 1.8. A group $(G, \odot)$ is a set G with a binary operation $\odot : G \times G \to G$ that has the following three properties:

1. The operation $\odot$ is *associative*, i.e. $(a \odot b) \odot c = a \odot (b \odot c)$ for all $a, b, c \in G$.
2. There exists a *neutral element* $e \in G$ such that $e \odot a = a \odot e = a$ for all $a \in G$.
3. For every element $a \in G$ there exists an element $b \in G$ such that $a \odot b = b \odot a = e$; this element b is called the *inverse* of a.

You have likely run into groups in the past, and a few examples might help to fresh up your memory:

- $(\mathbb{Z}, +)$, the integers with as operation addition. Its neutral element is 0 and the inverse of any $k \in \mathbb{Z}$ is given by $-k$.
- $(\mathbb{Z}/n, +)$, with $n \in \mathbb{N}$: the integers modulo n, with the operation addition modulo n. The neutral element is again 0, and the inverse of an $k \in \mathbb{Z}/n$ is given by $-k \pmod n$.
- $(\mathbb{Q}^*, \cdot)$, where $\mathbb{Q}^*$ denotes $\mathbb{Q} \setminus \{0\}$, is also a group with multiplication as operation. The neutral element is again 1 and the inverse of $\frac{n}{m}$ is $\frac{m}{n}$.
- $(\mathbb{Z}^2, +)$, the ordered integer pairs with vector addition, has neutral element $(0, 0)$; the inverse of (k, l) is simply $(-k, -l)$.

We will often write G instead of $(G, \odot)$; from the context it will follow which operation we have in mind. We can also make new groups out of old ones by selecting only a subset. These new groups are called subgroups:

Definition 1.9. A *subgroup* of a group $(G, \odot)$ is a non-empty set $H \subset G$ such that:

1. The binary operation $\odot$ restricted to H is closed, i.e. for all $a, b \in H$ we have $a \odot b \in H$.
2. The inverse of any element in H is again in H.

You can easily check that the neutral element $e \in H$ and that $(H, \odot)$ is a group again.

1.4 Symmetry Groups

Let's get back to our tilings and symmetries. We said that the symmetries of a set $S \subset \mathbb{R}^2$ or tiling $\mathcal{T}$ do form a group. By the following theorem this is indeed the case, under suitable conditions, hence we speak of $\mathcal{I}(S)$ and $\mathcal{I}(\mathcal{T})$ as the *symmetry group* of S, respectively $\mathcal{T}$.

Theorem 1.10. *Under any of the following conditions, the symmetries of a collection $\mathcal{C}$ of subsets of $\mathbb{R}^2$ do form a group w.r.t. to the operation of composition:*

(i) $\mathcal{C}$ is finite, i.e. contains finitely many elements (particular case: $\mathcal{C} = \{S\}$, $S \subset \mathbb{R}^2$), and

(ii) $\mathcal{C}$ is a tiling whose tiles have non-empty interior.

Proof. We start by showing that $\mathcal{I}(\mathbb{R}^2)$ is a group, and then we will show that under any of the above conditions, $\mathcal{I}(\mathcal{C})$ forms a subgroup of $\mathcal{I}(\mathbb{R}^2)$.

Remember the group operation is simply composition: A after B, denoted by $A \circ B$ or AB for short. In the first place we note that the composition of two symmetries of $\mathbb{R}^2$ is a symmetry again. Associativity is a property of composition; in general

$$((f \circ g) \circ h)(x) = (f \circ g)(h(x)) = f(g(h(x))) = f((g \circ h)(x)) = (f \circ (g \circ h))(x).$$

The neutral element of $(\mathcal{I}(\mathbb{R}^2), \circ)$ is the identity map Id. Because symmetries are isometries, they are bijective (cf. Corollary 1.5), and hence there exists for every $\sigma \in \mathcal{I}(\mathbb{R}^2)$ an inverse σ^{-1} such that $\sigma^{-1}\sigma = \sigma\sigma^{-1} = \mathrm{Id}$.

Now we show that under condition (i) or (ii), $\mathcal{C}$ is a subgroup of $\mathcal{I}(\mathbb{R}^2)$. If $\sigma_1, \sigma_2 \in \mathcal{I}(\mathcal{C})$, then for any $S \in \mathcal{C}$ we have, by applying the definition of symmetry to σ_2, that $\sigma_2 S \in \mathcal{C}$. By also applying this definition to σ_1, this is followed by the conclusion that $\sigma_1\sigma_2 S \in \mathcal{C}$. But then $\sigma_1\sigma_2 \in \mathcal{I}(\mathcal{C})$, or in other words: $\circ$ restricted to $\mathcal{I}(\mathcal{C})$ is closed.

To see that the inverse of any $\sigma \in \mathcal{I}(\mathcal{C})$ is again in $\mathcal{I}(\mathcal{C})$, we really need a condition like (i) or (ii). Consider a symmetry $\sigma \in \mathcal{I}(\mathcal{C})$ with inverse $\sigma^{-1} \in \mathcal{I}(\mathbb{R}^2)$.

Assume first that (i) holds. Consider the assicated map $\sigma^* : \mathcal{C} \to \mathcal{C}$ defined by $\sigma^* S = \sigma S$ for all $S \in \mathcal{C}$. (This definition is valid by the definition of a symmetry.) Note that if we have any $S, T \in \mathcal{C}$ with $S \neq T$, then also $\sigma S \neq \sigma T$ because σ is bijective. This shows that σ^* is an injective map. Since σ^* operates on a finite set, $\mathcal{C}$, it then also is bijective. Hence for every $S \in \mathcal{C}$ there exists a $T \in \mathcal{C}$ such that $S = \sigma^* T = \sigma T$, showing that σ^{-1} is indeed a symmetry of $\mathcal{C}$.

Assume now that (ii) holds. Let $S \in \mathcal{C}$ be chosen arbitrary, then we must show that $\sigma^{-1} S \in \mathcal{C}$. By assumption, S has non-empty interior, S°. Because $\mathcal{C}$ is a tiling, S° is disjunct with the interiors of the other tiles in $\mathcal{C}$. Since $(S^\circ)^c$ is closed, we know it contains the closures of the interiors of all the other tiles in $\mathcal{C}$. These closures simply equal the tiles themselves, hence S° is not only disjunct with the interiors of all the other tiles, but also with the other tiles themselves.

Now look at $\sigma^{-1}(S^\circ)$; because $\mathcal{C}$ is a tiling, this set intersects at least some tile $T \in \mathcal{C}$. The image of this tile T under σ now intersects S°. Because σ is a symmetry of $\mathcal{C}$, $\sigma T \in \mathcal{C}$. The only tile covering S° is S, hence from $\sigma T \cap S^\circ \neq \emptyset$ it follows that $\sigma T = S$. ☐

We intend to classify tilings according to their symmetry groups. This requires us to define when we consider two symmetry groups to be 'equal', or 'of the same type'. The concept of an isomorphism comes into play here:

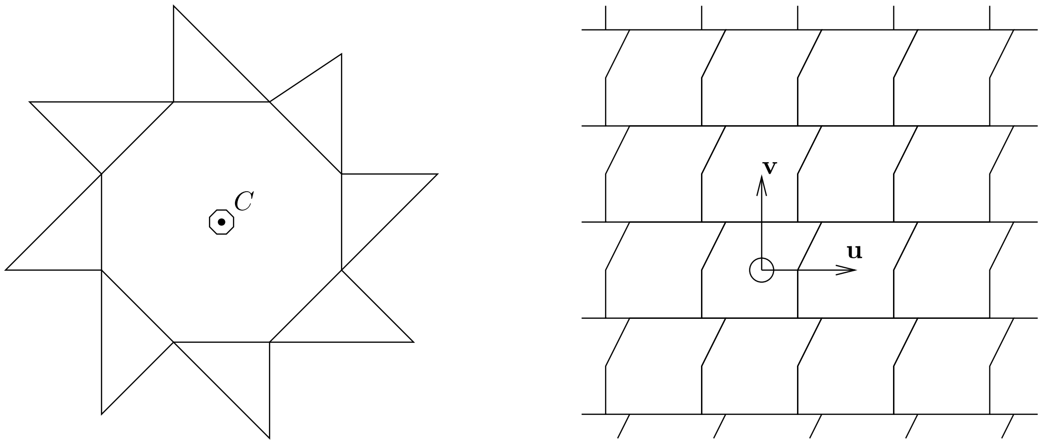

Figure 2: Collections having a symmetry group isomorphic to $(\mathbb{Z}/8, +)$ and $(\mathbb{Z}^2, +)$, respectively.

Definition 1.11. An *isomorphism* of two groups $(G, \odot)$ and $(H, \boxdot)$ is a bijective map $\phi : G \to H$ such that for all $a, b \in G$ we have $\phi(a) \boxdot \phi(b) = \phi(a \odot b)$.

Note that then also for all $a, b \in H$ we do have

$$\phi^{-1}(a \boxdot b) = \phi^{-1}\left(\phi\phi^{-1}(a) \boxdot \phi\phi^{-1}(b)\right) = \phi^{-1}\phi\left(\phi^{-1}(a) \odot \phi^{-1}(b)\right) = \phi^{-1}(a) \odot \phi^{-1}(b).$$

It seems tempting to define that we consider two symmetry groups to be of the same type if there simply exists an isomorphism between two, but we will require even a little bit more:

Definition 1.12. We consider two symmetry groups $\mathcal{I}(\mathcal{C}_1)$ and $\mathcal{I}(C_2)$ to be *of the same type* when there exists an isomorphism $\phi : \mathcal{I}(\mathcal{C}_1) \to \mathcal{I}(\mathcal{C}_2)$ that also maps the direct symmetries onto the direct symmetries: $\phi(\mathcal{I}(\mathcal{C}_1)^+) = \mathcal{I}(\mathcal{C}_2)^+$.

We consider some examples of symmetry groups.

- The symmetry group $\left\{R_{C,2\pi/k} : k = 0, \ldots, n-1\right\}$, for $n \geq 2$. It arises for example as the symmetry group of the collection of sets in the plane on the left in Figure 2 (with $n = 8$). This symmetry group is isomorphic with $(\mathbb{Z}/n, +)$.
- The symmetry group $\{T_{n\mathbf{u}+m\mathbf{v}} : n, m \in \mathbb{Z}\}$, with $\mathbf{u}$ and $\mathbf{v}$ two independent vectors in $\mathbb{R}^2$. It arises for example as the symmetry group of the tiling on the right in Figure 2 (with $\mathbf{u}$ and $\mathbf{v}$ the standard base vectors of $\mathbb{R}^2$). This symmetry group is isomorphic with $(\mathbb{Z}^2, +)$.
- The symmetry groups $\{\text{Id}, M_L\}$ and $\{\text{Id}, R_{C,\pi}\}$ are both isomorphic to $(\mathbb{Z}/2, +)$. However, they are *not* of the same type, as the former contains an indirect symmetry and the latter not. In fact, we made the special notion in Definition 1.12 above regarding the mapping of direct symmetries onto direct symmetries in order to be able to distinguish between these two symmetry groups.

As an additional remark, we note that the subset of direct symmetries $\mathcal{I}(\mathcal{C})^+$ of a symmetry group $\mathcal{I}(\mathcal{C})$ is a group again. We leave it to the reader to proof this simple fact.

1.5 Strip Groups

We now come to the classification of symmetry groups that are of the same type. Clearly there is an infinite number of symmetry groups containg no translations at all. For example, we can extending the figure on the left in Figure 2 to include an arbitrary number of rotations. Also, we will look only at the symmetry groups of 'nice' tilings, because otherwise the translations in the symmetry group might not constitute a translation lattice. To this end, it suffices that the tiles

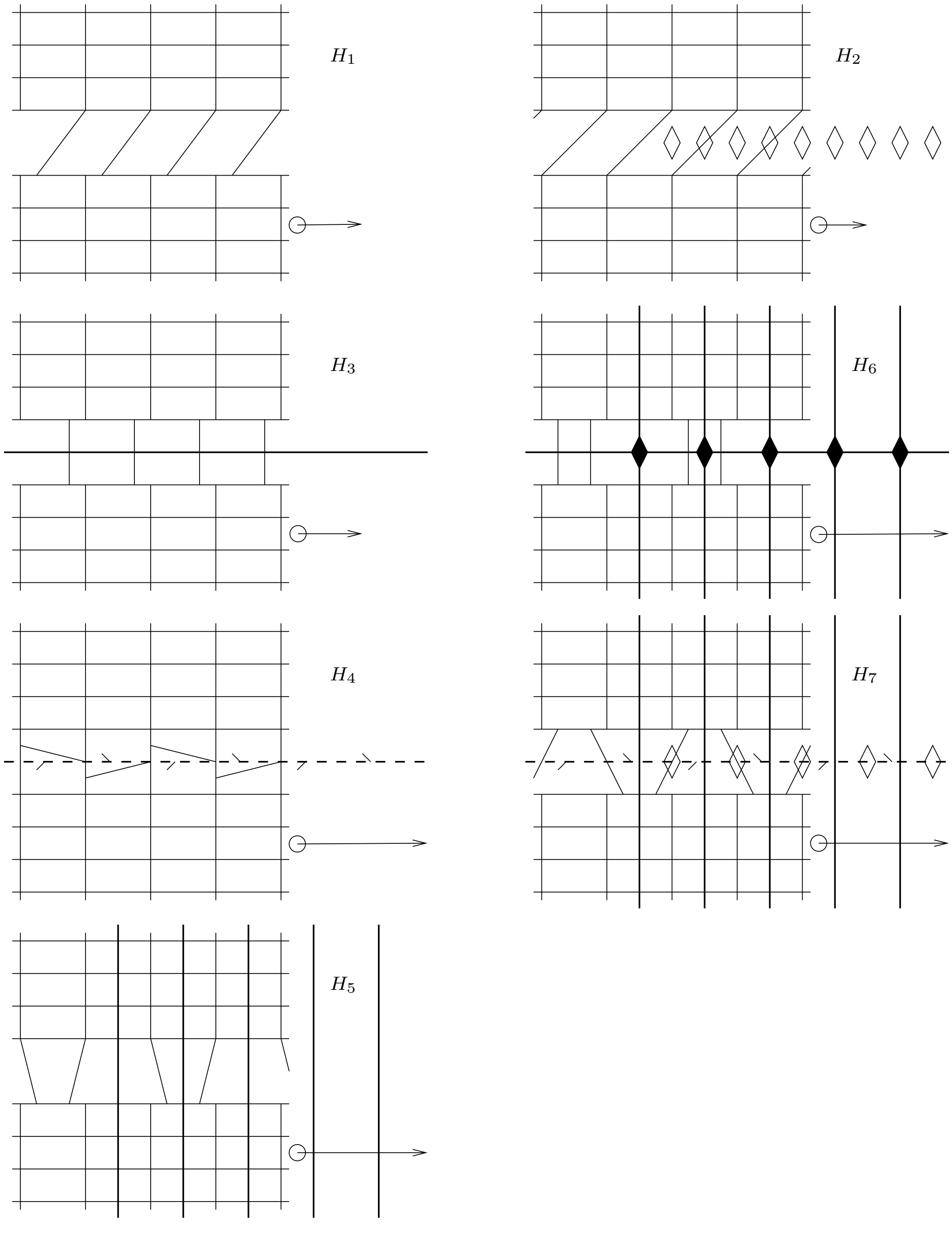

Figure 3: The seven strip groups.

1. are compact, and

2. have non-empty interior.

As shown before, the second condition garantuees that the symmetries actually do form a group. Note that topological disks are compact (as the closed unit disk is compact and homeomorphism preserve compactness), so together with Lemma 1.2 it follows that tilings of topological disks satisfy the above requirements. Also note that for $\mathbb{R}^2$, compactness of tiles is equivalent to boundedness.

As said, we will focus our attention to symmetry groups of tilings with bounded tiles having non-empty interior that do contain translations. In this section we will show that there are 7 so called *strip groups* — groups of such tilings whose translations are all parallel to one direction — and in the next section we will show that there are 17 so called *crystallographic groups* — groups of such tilings including translations in two independent directions, i.e. belonging to periodic tilings.

First a little word on notation: we will write for any symmetry A and $k \geq 1$:

$$A^k := \underbrace{A \circ A \circ \ldots \circ A}_{k \text{ times}}, \quad A^0 := \mathrm{Id} \quad \text{and} \quad A^{-k} := (A^{-1})^k.$$

With the *order* of a rotation $R_{C,\theta}$, we will mean the number

$$\inf \left\{k \in \mathbb{N}^* : R^k_{C,\theta} = R_{C,k\theta} = \mathrm{Id}\right\} = \inf \{k \in \mathbb{N}^* : (\exists l \in \mathbb{Z})\ k\theta = l2\pi\},$$

where $\mathbb{N}^*$ denotes the set of positive integers.

Let's start now with the strip groups. Consider a tiling $\mathcal{T}$ whose tiles are bounded and have non-empty interior. Now it can be shown that all translations are a multiple of one translation $T_{\mathbf{u}}$, i.e. the subset of translations in $\mathcal{I}(\mathcal{T})$ is given by $\left\{T^k_{\mathbf{u}} : k \in \mathbb{Z}\right\}$ for some $\mathbf{u} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$. We will call $T_{\mathbf{u}}$ the *generating translation* of the strip group. We leave it as an exercise to the reader to proof this. As a hint, consider the images under repeated translations $T_{\mathbf{u}}$ of a small open sphere that is completely contained within a tile. Deduce that there must be a 'smallest' $\mathbf{u}$ such that all translations are a multiple of $T_{\mathbf{u}}$.

We can now make the following important observation:

Lemma 1.13. *A strip group contains rotations of at most order 2.*

Proof. If a strip group would contain a rotation $R_{C,\theta}$ of an order larger than 2, then θ is not an integer multiple of π. Now, if the translation $T_{\mathbf{u}}$ is in the symmetry group, also $R_{C,\theta} T_{\mathbf{u}} {R_{C,\theta}}^{-1}$ is in the symmetry group. By geometrical construction, you can see that this symmetry is in fact a translation *not* parallel to $\mathbf{u}$, finishing the proof. □

In fact, if $\mathcal{I}(\mathcal{T})$ contains one rotation of order 2, it contains a lot more rotations of this order. This can be shown by combining the right translations and rotations: $T^k_{\mathbf{u}} R_{C,\pi} = R_{C+k\mathbf{u}/2,\pi}$ for all $k \in \mathbb{Z}$. Morover, $R_{C,\pi} R_{D,\pi} = T_{2(D-C)}$, showing that all centers of rotation must lie on a line at distances $\frac{1}{2}\|\mathbf{u}\|_2$. Together, his shows that the direct symmetries of a strip group are of one of these two forms:

- $H_1 = \{T_{k\mathbf{u}} : k \in \mathbb{Z}\}$, or
- $H_2 = \{T_{k\mathbf{u}} : k \in \mathbb{Z}\} \cup \left\{R_{C+k\mathbf{u}/2,\pi} : k \in \mathbb{Z}\right\}$.

In fact, H_1 and H_2 are symmetry groups themselves, see the corresponding tilings in Figure 3. We shall construct the five other strip groups using these two base groups.

So let's start with H_1, having generating translation $T_{\mathbf{u}}$. We can now add reflections and/or glide reflections to it. Note that any reflection added should either be in the direction of the generating translation $T_{\mathbf{u}}$, or be perpendicular to it, because otherwise a translation in another direction than $\mathbf{u}$ would appear.

With regard to glide reflections, we can say that appyling a glide reflection $G_{L,d}$ two times yields a translation along L over distance $2d$. Hence any glide reflection (that is not just a reflection) in the strip group must be parallel to $\mathbf{u}$. Moreover, there cannot be glide reflections in two different parallel lines in it, because together they would produce a translation having not only a component in the direction of $\mathbf{u}$, but also in the direction perpendicular to $\mathbf{u}$. Note that this not only is true for two glide reflections along two different parallel lines, but also for the special cases where one or both is a normal reflection. Also note that not both a (glide) reflection in a line parallel to $\mathbf{u}$ and a reflection in a line perpendicular to $\mathbf{u}$ can be in the symmetry group, as this would result in a rotation of order 2.

Let us add a glide reflection $G_{L,d}$, with L parallel to $\mathbf{u}$. Let $k \in \mathbb{Z}$ be such that $G_{L,d}^2 = T_{ku}$. We cannot add another reflection in a line perpendicular to $\mathbf{u}$, as discussed above. Also we find that $d = \frac{1}{2}\|k\mathbf{u}\|_2 = \frac{1}{2}k\|\mathbf{u}\|_2$. So all glide refections in L over a distance of the form $d + n\|\mathbf{u}\|_2 = (\frac{1}{2}k + n)\|\mathbf{u}\|_2$, with $n \in \mathbb{Z}$, are in the group. In particular, if k is even, then M_L is in the group. If k is odd, then surely M_L is not in the group, as then $T_{\mathbf{u}/2}$ would be in the group.

If there is no reflection in the group, then certainly M_L is not and thus, by the above, k is odd. But then all glide reflections in L over a distance of $(\frac{1}{2} + n)\|\mathbf{u}\|_2$ with $n \in \mathbb{Z}$ are in the group. If there is a reflection in the group, then it must be M_L (by statements made above), and now k must be even. Then all glide reflections in L over a distance of $n\|\mathbf{u}\|_2$ with $n \in \mathbb{Z}$ are in the group.

Actually, this shows that there are precisely two strip groups based on H_1 with glide reflections in it:

- $H_3 = \{T_{k\mathbf{u}} : k \in \mathbb{Z}\} \cup \{G_{L,kd} : k \in \mathbb{Z}\} \ni M_L$, and
- $H_4 = \{T_{k\mathbf{u}} : k \in \mathbb{Z}\} \cup \{G_{L,(k+1/2)d} : k \in \mathbb{Z}\} \not\ni M_L$,

where $d = \frac{1}{2}\|\mathbf{u}\|_2$ and $L \parallel \mathbf{u}$.

We can of course also add lines of reflection perpendicular to $\mathbf{u}$. If L is a line perpendicular to $\mathbf{u}$, then $T_{\mathbf{u}}^k M_L = M_{L+\mathbf{u}/2}$, another reflection in a line perpendicular to $\mathbf{u}$. If there are two reflections in lines perpendicular to $\mathbf{u}$, say M_K and M_L, then their product is a translation in the direction of $\mathbf{u}$, over a distance double the distance between K and L. Hence the distance between K and L must be half an integer multiple of $\|\mathbf{u}\|_2$. Hence there is only one group based on H_1 left:

- $H_5 = \{T_{k\mathbf{u}} : k \in \mathbb{Z}\} \cup \{M_{L+k\mathbf{u}/2} : k \in \mathbb{Z}\}, \quad L \perp \mathbf{u}$.

The groups based on H_2 can be found in a similar fashion. We won't get into all the details here. You can reuse some observations made in our discussion of augmenting H_1. The resulting groups are

- $H_6 = \bigcup_{k\in\mathbb{Z}} \{T_{\mathbf{u}}^k, R_{C+k\mathbf{u}/2,\pi}, G_{L,kd}, M_{K+k\mathbf{u}/2}\} \ni M_L$, and
- $H_7 = \bigcup_{k\in\mathbb{Z}} \{T_{\mathbf{u}}^k, R_{C+k\mathbf{u}/2,\pi}, G_{L,(k+1/2)d}, M_{K+(k+1/2)\mathbf{u}}\} \not\ni M_L$,

where $d = \frac{1}{2}\|\mathbf{u}\|_2$ and $K \perp L \parallel \mathbf{u}$.

1.6 Crystallographic Groups

We will now briefly consider the crystallograhpic groups of a tiling $\mathcal{T}$. Note that by assumption the tiles of $\mathcal{T}$ are bounded and have non-empty interior, and $\mathcal{I}(\mathcal{T})$ contains translations in two independent directions.

It can be shown that under these assumptions, there exists independent vectors $\mathbf{u}$ and $\mathbf{v}$ such that all translations in $\mathcal{I}(\mathcal{T})$ are given by $\{T_{n\mathbf{u}+m\mathbf{v}} : n, m \in \mathbb{Z}\} = \{T_{\mathbf{u}}^n \circ T_{\mathbf{v}}^m : (n, m) \in \mathbb{Z}^2\}$. Analogous to the strip case, $T_{\mathbf{u}}$ and $T_{\mathbf{v}}$ are called the generating translations.

The proof of this is again left as an exercise, with a similar hint as in the strip case: consider the images of a small sphere contained within a given tile under two arbitrary 'smallest' independent translations $T_{\mathbf{u}}$ and $T_{\mathbf{v}}$ that are in the symmetry group. Map all images to the parallellogram spanned by $\mathbf{u}$ and $\mathbf{v}$ in a modulo-like way. Deduce that there cannot be another translation $T_{\mathbf{w}}$ in the symmetry group with $\mathbf{w}$ not an integer linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Associated with every point P in the plane, there is a *translation lattice* consisting of the images of that point under the translations in $\mathcal{I}(\mathcal{T})$, i.e. the lattice is given by $\{T_{\mathbf{u}}^n \circ T_{\mathbf{v}}^m(P) : (n, m) \in \mathbb{Z}^2\}$.

Now we can look at the Voronoi cells formed by the translation lattice of a point P. A *Voronoi cell* $V(Q)$ of the point Q in the lattice (of a point P) is defined as

$$V(Q) = \{X \in \mathbb{R}^2 : \mathrm{d}(X,Q) \leq \mathrm{d}(X,R) \text{ for all lattice points } R \neq Q\}.$$

Thus, a Voronoi cell constitutes the set of all points that have Q closest to it among all lattice points.

There exist a number of theorems about the shape of these cells that will help classifying the crystallographic groups. We will not prove them, but just mention them as a number of important observations. In the following statements we consider the lattice belonging to a point P.

- First, we note that all Voronoi cells of a lattice are congruent, i.e. they have the same shape; moreover they have the same orientation.
- Secondly, when looking at a point Q in the lattice, the shape of the boundary of the Voronoi cell $V(Q)$ belonging to Q, is either a rectangle or a hexagon, symmetric with respect to Q (i.e. $R_{Q,\pi}$ is a symmetry of $V(Q)$).
- Lastly, a rotation in the symmetry group of $\mathcal{T}$, not equal to Id, with its center at one of the lattice points Q, maps the lattice onto itself. This implies that the rotation maps the Voronoi cell $V(Q)$ of Q onto itself. By using the possible shapes of the Voronoi cell, we conclude that the rotation has either order 2, 3, 4 or 6.

Now, we can build the 'basic' symmetry groups consisting of only direct symmetries by allowing rotations of increasing order to appear. This approach yields five symmetry groups, which we will denote by G_1 through G_5.

From these five groups, the twelve other groups can be constructed. It would be way out of the scope of this lecture to fill in all the details for that construction. There are numerous websites on the internet that list these symmetry groups, together with examples of tilings or patterns that having these symmetry groups and even with animations showing the corresponding symmetries.

Finally, we can make one last remark about our definition of symmetry groups begin of the same type. In this definition, we demanded that the direct symmetries were mapped onto the direct symmetries by the isomorphism involved. For the strip groups, this requirement cannot be dropped, as then H_2 and H_5 turn out to be isomorphic, but $H_2^+ \neq H_5^+$. For the crystallographic groups, this requirement *can* be dropped however! This, in fact, is a theorem that can be proven by showing that none of the groups G_1 through G_{17} is isomorphic to another. See e.g. [A], which elaborates on this whole subject.

References

[GS] Grünbaum, Branko, and Shephard, G.C. – Tilings and patterns. An introduction. W.H. Freeman and Company, New York, 1989.

[A] Aarts, J.M. – Meetkunde (facetten van de planimetrie en stereometrie). Epsilon Uitgave, Utrecht, 2000. *Note: an English translation of this book might become available.*